ENTANGLING QUBIT WITH THE REST OF THE WORLD – THE MONOGAMY PRINCIPLE IN ACTION

Summary. A simple derivation of the finite Schmidt decomposition of the pure states describing finite dimensional systems interacting with the infinite dimensional one is presented. In particular, maximally entangled pure states in such systems are being characterized.

Keywords: quantum entanglement, monogamy principle, Schmidt decomposition, spin-orbit entanglement

1. Introduction

Quantum correlations contained in quantum entangled states describing composite quantum systems are by no doubts one of the major resources for several quantum
information tasks [1-5]. One of the very interesting feature of the quantum entanglement is its monogamy nature which means, roughly, that a total available amount of quantum correlations contained in quantum states is always of a limited capacity, depending on the very nature of system considered. In particular, there exist quantum states containing maximal possible amount of quantum correlations and exactly these states are often most wanted for performing several quantum operations like: teleportation of states, cryptographic protocols implementations and many others. This is the main reason that the mathematical description and the corresponding engineering technologies of preparing physically such maximally entangled states seems to be of great importance.

In the case of composite systems being in the maximally entangled state it is impossible further to entangle such quantum system with another quantum system, this monogamy principle is the major element that defends the security of the most of the implemented technically cryptographic protocols up to date [6].

In the case of finite dimensional systems a lot of work has been done on the very nature of quantum entanglement [1-4], the case of two-partite systems is the best recognised situation. In the case of two-partite, finite dimensional systems, the Schmidt decomposition of the corresponding pure state gives essentially all relevant information on the corresponding quantum correlations. From the quantitative point of view and from qualitative ((S)LOCC type, semiorder relations [2-4]) point of view as well. The case of many-partite systems, and also the case of mixed (even for two-partite systems) states is much less recognised despite to many efforts [2-5].

The case of two-partite systems composed from finite dimensional system coupled with some infinite dimensional one is being discussed in the present note. A simple construction of the finite Schmidt decomposition of the pure states of such systems will be demonstrated here and some straightforward consequences of the derived decomposition will be formulated. A particular example of a single qubit coupled to the rest of the World will be presented as an example. In particular, a general form of maximally entangled pure states of such systems is being derived and the corresponding amount of entanglement contained and defined as the von Neumann entropy of the arising reduced density matrices is calculated.

The present day physics od spin-orbit entanglement effects is the main motivation for our material presented in this note. Spin-orbit (SO) coupling – the interaction between a quantum particle’s spin and its momentum – is ubiquitous in physical systems. The role playing in the atomic and molecular systems by the spin – orbit coupling is very well known since the very beginning of quantum theory of matter [7, 8]. In recent years some new effects of spin – orbit correlations have been observed, the very nature of which can be explained on the assumption on the presence of entanglement between spin degrees of freedom with the orbital part of the
corresponding wave functions. For more information on this and references see [9-12]. Also in condensed matter systems, SOE coupling and entanglement is crucial for the spin-Hall effect and topological insulators and is important for spintronic devices which might to appear to be of major importance in the future quantum computers hardware industries. Quantum many-body systems of ultracold atoms forming new state of matter, the so called Bose-Einstein condensates can be today precisely controlled experimentally, and would therefore to provide an ideal platform on which one can study SOE coupling effects [9-12].

The organization of the material is the following one. In the next section we present very simple construction of the finite Schmidt decomposition of pure states in the \((d, \infty)\) class of two-partite systems. In particular the maximally entangled pure states of such systems have been described. The particular example of a single qubit entangled with the rest of the world described by some separable Hilbert space is discussed in more details in the final section.

2. The Schmidt decomposition theorem for pure states of \((d, \infty)\) systems

Let \(h_1, h_2\) be a pair of separable Hilbert spaces, then the tensor product space \(H = h_1 \otimes h_2\) is again separable Hilbert space with the corresponding scalar product

\[
< - | >_H = < - | >_{1}^* < - | >_{2}
\]

for product vectors in \(H\) and then extended by linearity and continuity to the whole of \(H\).

The set of pure states \(\partial E(H)\) on the space \(H\) is defined as the stratification of the action of unitary group \(U(1)\) (the global phase calibration) on the unit sphere in \(H\). The space of quantum states \(E(H)\) on \(H\) is defined as the intersection of the cone of nonnegative, trace class of operators with the hyperplane \(tr(\rho)=1\). From now on, we shall to assume that the factor \(h_1\) is finite dimensional space, i.e. without loose of generality we assume that \(h_1\) is the \(d\)-dimensional space \(C^d\) with the standard Hilbert space structure.

For a given, separable Hilbert space \(H\), we denote by \(\text{CONS}(H)\) the space of complete orthonormal systems in \(H\).

Let \(H = h_1 \otimes h_2\), \(\text{dim}(h_1) = d < \infty\) and let \(\{e_\alpha: \alpha = 1: d\} \in \text{CONS}(h_1), \{f_\kappa: \kappa = 1, \ldots\} \in \text{CONS}(h_2)\). Then the set \(\{e_\alpha \otimes f_\kappa, \alpha = 1: d, \kappa = 1, \ldots\}\) forms a complete orthonormal system in \(C^d \otimes h_2\). Therefore, taking any vector \(\Psi \in H\) we have the following decomposition (norm convergent):

\[
\Psi = \sum_{\alpha, \kappa} \psi_{\alpha, \kappa} e_\alpha \otimes f_\kappa
\]

where: \(\psi_{\alpha, \kappa} = < \Psi | e_\alpha \otimes f_\kappa >\) and \(\sum_{\alpha, \kappa} |\psi_{\alpha, \kappa}|^2 = \|\Psi\|^2\)
Let us define the following linear map:

$$J(\Psi) : C^d \rightarrow h_2$$  \hfill (2.2)

$$\sum_{a=1}^{d} c_a \ast e_a \rightarrow \sum_{a=1}^{d} c_a \ast J(\Psi) (e_a) =$$

$$\sum_{a=1}^{d} c_a \sum_{j=1}^{\aleph} \psi_{a,j} \mid f_j >$$

$$\sum_{j=1}^{\aleph} (\sum_{a=1}^{d} c_a \ast \psi_{a,j}) \mid f_j >$$

With the use of the Cauchy-Schwartz inequality it is not difficult to prove that the map $J(\Psi)$ is a bounded linear map and:

$$\| J(\Psi) \|_{L(C^d \rightarrow h_2)} = \sup_{\| x \|_{C^d}} \| J(\Psi)(x) \| \leq \| \Psi \|$$  \hfill (2.3)

Therefore the conjugate map:

$$\| J^* (\Psi) : h_2 \rightarrow C^d$$

is defined well by the identity:

$$< F \mid J(\Psi)x >= < J^* (\Psi)F \mid x >$$

for all $x \in C^d$ and $F \in h_2$.

Defining:

$$\Delta(\Psi) = J^* \ast J(\Psi) : C^d \rightarrow C^d$$  \hfill (2.4)

it follows easy

(i) $\Delta(\Psi) = \Delta^* (\Psi) \geq 0$

(ii) $\| \Delta(\Psi) \|_{s_{\Psi}} = tr_{\gamma} \Delta(\Psi) = \| \Psi \|$.

From the spectral theorem for hermitean matrices it follows the following spectral decomposition of the matrix $\Delta(\Psi)$:

$$\Delta(\Psi) = \sum_{a=1}^{d} \tau_a \ast | f_a > < f_a |$$  \hfill (2.5)

where $\tau_a \geq 0$ are the the corresponding eigenvalues of $\Delta$ and $f_a$ are the corresponding eigenvectors. It also assumed the the appriopriate orthonormalisation procedura is performed (in the case of degenerated eigenvalues) that leads to orthogonality of the system of eigenvectors $f_a$.

The operator $J(\Psi) : C^d \rightarrow h_2$ being bounded is automatically closed and therefore the well known polarisation decomposition theorem can be applied [13, 14] as well. This leads to the existence of some isometric map $V:( Ker( J(\Psi)) )^\perp \rightarrow Rank( J(\Psi))$ and such that the following equality holds:
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\[ J(\Psi) = V \sqrt{\Delta(\Psi)}. \]  

(2.6)

Using the decomposition (2.4) and comparing with the definition (2.2) we are seeing that:

\[ \Psi = \sum_{a=1}^{d} \tau_a | f_a > \langle g_a | \]  

(2.7)

where \( g_a = V | f_a > \).

Thus we have proved the following result.

**Theorem (Schmidt decomposition theorem)**

Let \( H = \mathbb{C}^d \otimes h \), where \( h \) is a separable Hilbert space. Then, for any \( \Psi \in H \) there exists an unique sequence of numbers \( \tau_a, a = 1:d \) and two CONS \( \{ f_a \} \) in \( \mathbb{C}^d \) and \( \{ g_k \} \) in \( h \) such that:

\[ \Psi = \sum_{a=1}^{d} \tau_a | f_a > \langle g_a | \]  

(2.8)

The crucial point here is that the corresponding Schmidt decomposition is finite, in particular, for any \( \Psi \) the Schmidt rank is always finite and its maximal possible value is equal to \( d \). The coefficients \( \tau_a \) defined in the unique way for a given \( \Psi \) are called Schmidt coefficients of \( \Psi \).

A straightforward consequence of this decomposition Theorem is the following result (being well known in the case of \((d, d')\)-systems [15]).

**Corollary 1**

Let \( H = \mathbb{C}^d \otimes h \), where \( h \) is a separable Hilbert space and \( d < \infty \). Then for any vector \( \Psi \in H \) the reduced density matrix \( \rho_{c'}(\Psi) \) of \( \Psi \) after tracing out the degrees of freedom contained in \( h \) are given by the formula:

\[
\rho_{c'}(\Psi) = tr_h(| \Psi > \langle \Psi |) = \begin{bmatrix}
\tau_1^2 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \tau_d^2
\end{bmatrix}
\]  

(2.9)

where \( \tau_a \) are the corresponding Schmidt coefficients of the vector \( \Psi \).

Among \( \tau_a \) different notions [2-5] of quantitative measures of amount of entanglement contained in pure quantum states we propose here, for an illustrative purposes mainly, the standard entropic measure \( \text{Ent} \) based on the notion of von Neumann entropy of the density matrix:

\[
\text{Ent}(\Psi) = -tr \rho_{c'}(\Psi) \ln(\rho_{c'}(\Psi))
\]  

(2.10)

Using the result (2.9) we have:
\[ \text{Ent}(\Psi) = -\sum_{\alpha=1}^{d} \tau_{\alpha}^2 \# \ln \tau_{\alpha}^2. \]  

(2.11)

As is well known the maximum value of this function is attained for the case \( \tau_{\alpha} = 1/\sqrt{d} \) for all \( \alpha \). In this way we have proved the following Corollary.

**Corollary 2**

Let \( H = C^d \otimes h \), where \( h \) is a separable Hilbert space and \( d < \infty \). Any maximally entangled pure state \( \Psi \) of the system described by \( H \) is necessary of the form:

\[ \Psi = \sum_{\alpha=1}^{d} \frac{1}{\sqrt{d}} \psi_{\alpha} \otimes f_{\alpha} \]  

(2.12)

where \( \{ \psi_{\alpha}, \alpha = 1:d \} \in \text{CONS}(C^d) \) and \( \{ f_{\alpha}, \alpha = 1:d \} \) is \( d \)-dimensional orthonormal system of vectors in \( h \).

These results can also be extended to the case of the spaces \( L(C^d) \otimes h \), and also to the more interesting situation of \( C^d \otimes L_2(h) \) space, here \( L_2(h) \) refers to the Hilbert-Schmidt structure on the space \( L(h) \) of bounded, linear operators on \( h \). Details of these constructions together with some applications will be presented in a separate paper [16]. See also [13].

Bipartite, pure state entanglement manipulations are well understood presently in the case of bi-partite, finite-dimensional systems. In particular, the class of LOCC operations is coincident with the class SEP in this case [17]. Basing on the above mentioned fact, together with the theory of bistochastic matrices a complete description of the LOCC semiorder on the space of pure states can be translated into the corresponding semi-order relations on the space of the Schmidt coefficients as was shown by Nielsen [18] and developed further in many papers [2-4]. The very interesting problem here is to extend these results to the case of \((d, \infty)\) systems considered here, see [16].

The presented here derivation of the Schmidt decomposition is hardly to be used for the practical, even computer assisted calculations due to the a priori infinite series to be summed up (see equation (2.1)). However, in some very special situations the needed numerical calculations can be performed by standard, widely available tools like SVD decomposition. The one of such situation do appear if the vector \( \Psi \) is given by the finite sum of the form:

\[ \Psi = \sum_{a,k} e_{a,k} e_{a} \otimes f_k \]  

(2.13)

where \( e_{a} \in C^d \) and \( f_k \in h \). Let \( r = \text{rank}\{e_{a}\} \leq d \) and \( s = \text{rank}\{f_k\} \leq \infty \) be the ranks (i.e. dimensions of the linear subspaces in the corresponding vector spaces spanned by the corresponding sets of vectors) of the arising system of vectors in the representation (2.13). Then we can perform the Gram-Schmidt orthonormalisation operation obtaining a new systems \( \{ e^*_{\beta}, \beta = 1:r \} \) and
\{f^*_k, k=1:n\}, this time we obtain the systems that are composed from orthogonal and normalised vectors and the vector \(\Psi\) can be represented as:

\[
\Psi = \sum_{a,k} D_{a,k} e^*_{a} \otimes f^*_k
\]  

(2.14)

Completing (if necessary) the orthonormal systems used in (2.14) we can apply by suitable reorganisation the corresponding matrix \(D\) the standard SVD decomposition method to compute the Schmidt coefficients of the vector \(\Psi\).

The following Theorem may be used also to compute some approximations to the exact values of rank and Schmidt coefficients in the considered here case.

**Corollary 3**

Let \(\Psi_n\) be a norm convergent sequence of vectors in \(H=\mathbb{C}^d \otimes h\), where \(h\) is separable. Let \(SD = \{r_n, \{r^n_\alpha\}\}\) be the corresponding Schmidt data for \(\Psi_n\). Then

\[
\lim_{n \to \infty} r_n = r
\]

and

\[
\lim_{n \to \infty} r^n_\alpha = r_\alpha \quad \text{for } \alpha = 1: d
\]

where \(r, r_\alpha\) are the Schmidt data of \(\Psi\).

**Proof:**

From the linearity of the map \(J\) and the estimate (2.3) it follows that if \(\Psi_n \to \Psi\) in the norm as \(n \to \infty\) then \(J(\Psi_n) \to J(\Psi)\) in the norm. Therefore also \(\Delta(\Psi_n)\) tends to \(\Delta(\Psi)\) as \(n \to \infty\) in the norm topology. Thus, we have proved that:

\[
\lim_{n \to \infty} \Delta(\Psi_n) = \Delta(\Psi) \quad \text{in the norm of } L(C^d).
\]

Let us order the eigenvalues \(\{r^n_\alpha, \alpha=1:d\}\) of the matrix \(\Delta(\Psi_n)\) and of the matrix \(\Delta(\Psi)\) in the decreasing order. Then, by using the Weyl theorem [19] we have:

\[
\max_{1 \leq \alpha < d} |r^n_\alpha - r_\alpha| < ||\Delta(\Psi_n) - \Delta(\Psi)||.
\]

(2.16)

To apply this theorem to the computability of Schmidts data for a given vector \(\Psi\) given by the infinite dimensional representation like (2.1) we observe that any such vector can rewritten in the following form:

\[
\Psi = \sum_{a,k} \psi_{a,k} e^*_a \otimes f_k = \sum_k (\sum_{a} \psi_{a,k} e^*_a) \otimes f_k
\]

(2.17)

Let \(P_N\) be an orthogonal projection on the subspace \(h_N=1.h \{f_1, \ldots, f_N\}\) of \(h\). We define \(\Psi_N=P_N \Psi\), then \(\Psi_N\) is finite dimensional represented and \(\Psi_N \to \Psi\) as \(N \to \infty\) in the norm. Therefore, we can apply the standard SVD decomposition method computing the corresponding Schmidt data for the sequence of vectors \(\Psi_N\). Then we analyse behaviour of the Schmidt coefficients of vectors \(\Psi_N\) having hope to guess the corresponding limiting values.
As the corresponding sequences of Schmidt numbers are convergent as it follows from theorem 3. they are automatically Cauchy type sequences. In any case by this method we can compute the exact value of Schmidt rank of $\Psi$ and the Schmidt coefficients of $\Psi$ with an arbitrary precision in a finite time.

3. Single qubit entangled to the rest of the world

Let $d=2$. Then for any orthonormal system $\{e_1, e_2\}$ of vectors in $C^2$ and any $\text{CONS}\{f_k, \ k=1,2, \ldots \}$ in a separable Hilbert space $h$ and any normalised vector $\Psi \in C^2 \otimes h$ we can write:

$$\Psi = \sum_{a,k} \psi_{a,k} e_a \otimes f_k = c_1^* e_1 \otimes \theta_1 + c_2^* e_2 \otimes \theta_2$$  \hspace{1cm} (3.1)

where:

$$\theta_a = \sum_{k=1}^{\infty} \psi_{a,k}^* f_k^* \| \sum_{k=1}^{\infty} \psi_{a,k}^* f_k \|^{-1}$$  \hspace{1cm} (3.2)

for $a=1,2$, $c_a = \left( \sum_k |\psi_{a,k}|^2 \right)^{1/2}$.

The index of similarity $\delta$ is defined as:

$$\delta = \left| 1 - <\theta_1 | \theta_2 > \right|.$$

(3.3)

The case $\delta=1$

In this case $<\theta_1 | \theta_2 > = 0$ and then the expression (3.2) is exactly the Schmidt decomposition of and the corresponding entropy of entanglement contained in $\Psi$ is given by:

$$\text{Ent}(\Psi) = -c_1^2 \log(c_1^2) - c_2^2 \log(c_2^2)$$

(3.4)

which attains its maximum for $c_a = 1/\sqrt{2}$ and then $\text{Ent}(\Psi) = \log(2)$.

The case $\delta=0$

In this case the vectors $\theta_1$ and $\theta_2$ are linearly dependent and the vector $\Psi$ is separable.

The case $0<\delta<1$

In this case the vectors $\theta_1$ and $\theta_2$ are linearly independent and by applying the known Gram-Schmidt procedure it is possible to rotate the vectors $\theta_1$ and $\theta_2$ into the system of orthonormal vectors $\{\theta_1^*, \theta_2^*\}$ and such that:

$$\theta_1 = g_{11} \theta_1^* + g_{12} \theta_2^*$$

$$\theta_2 = g_{21} \theta_1^* + g_{22} \theta_2^*$$

(3.5)
with $|g_{a1}|^2 + |g_{a2}|^2 = 1$ for $a=1, 2$

and: $g_{11}=1$, $g_{12}=0$, $g_{21}=-\alpha/\beta$, $g_{22}=1/\beta$ where:

$$\alpha = \sqrt{\frac{1}{1-\sigma^2}}, \beta = \sqrt{\frac{1}{1-\sigma^2}}, \sigma = \langle \theta_1 | \theta_2 \rangle$$  \hspace{1cm} (3.6)

Therefore, we can rewrite $\Psi$ as:

$$\Psi = c_1 \* e_1 \otimes \theta_1^* + c_2 \* \sigma \* e_2 \otimes \theta_2^* + \frac{c_2}{\beta} \* e_2 \otimes \theta_2^*$$  \hspace{1cm} (3.7)

It is not difficult to compute the corresponding density matrix the qubit after tracing out the degrees of freedom corresponding to the environment in which the qubit is located:

$$\rho_{c^*} = tr_n <\Psi | \Psi> = \begin{bmatrix} c_1^2 & c_1 c_2 \sigma \\ c_1 c_2 \sigma^* & c_2^2 \end{bmatrix}$$  \hspace{1cm} (3.8)

By an elementary (but little bit tedious) calculations of the corresponding eigenvalues of the matrix $\rho_{c^*}$ it follows that for $1<|\sigma|<1$

$$Ent(\rho_{c^*}) = -tr \rho_{c^*} \ln(\rho_{c^*}) < \ln(2).$$ \hspace{1cm} (3.9)

Therefore if $0<\delta<1$, then always $Ent(\Psi)<\ln(2)$.

As the final conclusion we can formulate the following Monogamy Principle for a single qubit:

*If a single qubit is entangled with any quantum system and the whole composite system state is pure (or even mixed!, due to the convexity of entropy [20]) and the entropy of this entanglement is equal $\ln(2)$ then it is impossible to entangle this qubit with any other quantum system.*

**BIBLIOGRAPHY**


Omówienie

Bezpieczeństwo transferów kluczy w kanałach kwantowych, oparte na znanych protokołach kwantowych, w sposób istotny zależy od zjawiska splątania kwantowego, w szczególności własności monogamii tego splątania. Zasada monogamii splątania kwantowego, mówiąc obrazowo, głosi, że w wieloczęściowych układach kwantowych występuje zjawisko wysycenia poziomu korelacji kwantowych między różnymi jego częściami i niemożliwe staje się fizycznie dalsze jego splątanie kwantowo-mechaniczne.
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z innym układem kwantowym. Wysycenie poziomu korelacji kwantowych ma miejsce w stanach maksymalnie splatanych. Z tego powodu tak ważny jest opis matematyczny stanów maksymalnie splatanych różnych układów kwantowych oraz ich inżynieria związaną z ich fizyczną implementacją.

W artykule tym dyskutowane są dwuczęściowe układy kwantowe złożone z układu skończenie wymiarowego, kubitowego oraz układu nieskończenie wymiarowego, opisanego za pomocą ośrodowych przestrzeni Hilberta. Podano obliczeniowo dostępną (w sensie analizy numerycznej) wersje skończenie wymiarowego rozkładu dla Schmidta dla stanów czystych tej klasy układów i scharakteryzowano poziom jego splątania w terminach współczynników Schmidta tego rozkładu. W szczególności podana została kompletna charakteryzacja stanów czystych, maksymalnie splatanych opisujących układy tej klasy.

Jako przykład podano charakteryzacje stanów czystych, maksymalnie splatanych układu kwantowego złożonego z jednego kubitu oddziałującego z nieskończenie wymiarowym układem, np. z atomem lub molekulą. Otrzymane wyniki zostały zastosowane (w innej publikacji autora) do opisu matematycznego i ścisłej analizy spląć typu spin-orbita występujących często w różnego rodzaju układach atomowo-molekularnych i ciało-stałowych (a które to układy są bardzo poważnymi, na dzisiaj, kandydatami na udane implementacje technologii przetwarzania kwantowego danych).

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